Spring 2020 Qualifying Exam
OPTIMIZATION

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book your code number. *Do not write your name on any answer book.* On *one* of your books list the numbers of *all* the questions answered.
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer all 4 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. **Optimization Modeling**

The National Agricultural Bank (NAB) decides to establish an early retirement scheme for fifteen employees who are taking early retirement. These employees will retire over a period of seven years starting from the next year. To finance this early retirement scheme, the bank decides to invest in bonds for this seven-year period. The necessary amounts to cover the pre-retirement leavers are given below:

Year	1	2	3	4	5	6	7
Amount (1000 \$)	1000	600	640	480	760	1020	950

These amounts have to be paid at the beginning of every year, and year 1 starts right now, so we have an immediate obligation of \$1,000,000 to cover the employees taking early retirement at the start of the planning horizon.

For the investments, the bank decides to take three different types of bonds: SNCF bonds, Fujitsu bonds, and Treasury bonds. The money that is not invested in these bonds is placed as savings with a guaranteed yield of 3.2%. Information concerning the (annual) yield, durations and value of the bonds is given below:

Bond	Value (\$1000)	Interest Rate	Duration
SNCF	1.0	7.0 %	5 years
Fujitsu	0.8	7.0 %	4 years
Treasury	0.5	6.5 %	6 years

It is only possible to buy integer number of bonds and the invested capital remains locked in for the total duration of the bond. The capital invested in bonds is returned at the end of the bond's duration. Each year, the interest on the savings and the interest from the purchased bonds is returned. For example, suppose that you bought 10 Fujitsu bonds. This would cost you $10 \times \$800 = \$8000$ right now, and at the beginning of years 2, 3, 4, and 5, you would receive $7\% \times \$8000 = \$560$ in interest. Also in the beginning of year 5 (end of year 4), you would receive back the initial \$8000 that could be used to meet retirement obligations.

The person in charge of the retirement plan decides to buy bonds at the beginning of the first year, but not in the remaining years. How should she organize the investments in order to spend the least amount of money (right now) to cover the projected retirement plan?

(a) Write an optimization model that determines the optimal number of each bond type to buy and the amount to invest in the savings account in order to minimize the current investment and meet obligations. If you wish to write a general model, you may use the following notation:

- Time periods (years) $T = \{1, 2, \ldots, 7\}$
- Set of Bonds $B = \{\text{SNCF, Fujitsu, Treasury}\}$
- Obligation Amounts (\$): $h_t, \forall t \in T$
- Bond Prices/Value (\$): $p_b, \forall b \in B$
- Annual rate of return of bonds (%): $r_b, \forall b \in B$
- Duration of each bond (years): $d_b, \forall b \in B$
- Risk free rate of return (%): $\kappa = 3.2\%$.

(b) Suppose the bank could pay a bonus of 1% of the value additionally to retirees if they delay retirement by one year. For example, the bank could decide not to meet its current \$1,000,000 obligation to its retirees and instead pay these retirees $(1.01) \times \$1,000,000 = \$1,010,000$ at the start of year two. Retirees can only be deferred for 1 year. For instance, you cannot pay the people who want to retire now $(1.01) \times (1.01) \times \$1,000,000$ in two years. The people retiring at the beginning of year 7 cannot be deferred. Extend your model to determine if this saves money or not, and in which years the incentive should be offered? You may do this instance in the general model or modeling language. But be sure to clearly define any new decision variables and equations.

2. **Linear Optimization**

Consider the linear programming problem (P) defined as

$$\min_{x,M} \quad M \tag{1}$$
$$\text{s.t.} \quad \sum_{j=1}^{n} x_j = 1 \tag{2}$$
$$\sum_{j=1}^{n} a_{ij}x_j \leq M \qquad \text{for } i = 1,\dots,m \tag{3}$$
$$x_j \geq 0 \qquad \text{for } j = 1,\dots,n, \tag{4}$$

where a_{ij} is the ij-th entry of a given matrix $A \in \mathbb{R}^{m\times n}$.

(a) Denote by N and y_i, $i = 1,\dots,m$, the dual variables associated to the primal constraints (2) and (3), respectively. Write the dual (D) of (P) and bring it into a form where the dual vector $y \in \mathbb{R}^m$ is constrained to be nonnegative.

(b) Show that both (P) and (D) have a finite optimum that is attained at a vertex of their feasible sets.

(c) Prove that for each feasible solution (x, M) to (P) and for each feasible solution (y, N) to (D) it holds

$$N \leq \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} x_j \leq M.$$

(d) Let $\bar{x}$ and $\bar{y}$ be feasible solutions to (P) and to (D), respectively. Derive closed form solutions to the problems

$$\begin{aligned} \min_{x} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{y}_i a_{ij} x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j = 1 \qquad \text{(P')} \\ & x \geq 0, \end{aligned}$$

$$\begin{aligned} \max_{y} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} \bar{x}_j \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = 1 \qquad \text{(D')} \\ & y \geq 0. \end{aligned}$$

(e) Prove Von Neumann's Minimax Theorem for a two-person zero-sum game:

$$\max_{y\in Y}\min_{x\in X}\sum_{i=1}^{m}\sum_{j=1}^{n} x_j a_{ij} y_i = \min_{x\in X}\max_{y\in Y}\sum_{i=1}^{m}\sum_{j=1}^{n} x_j a_{ij} y_i,$$

where $X = \{x \in \mathbb{R}^n | x \geq 0, \sum_{j=1}^{n} x_j = 1\}$ and $Y = \{y \in \mathbb{R}^m | y \geq 0, \sum_{i=1}^{m} y_i = 1\}$.

3. **Integer Optimization**

Let $D = (V, A)$ be a complete digraph with n nodes, and denote by $\mathcal{C}$ the set of directed cycles in D with k nodes, for $k \in \{2, 3, \dots, n-2\}$. Let P be defined as the set of vectors $x \in \mathbb{R}^A$ satisfying the following constraints:

$$\sum_{ij \in A: j \neq i} x_{ij} = 1 \qquad \forall i \in V \qquad \text{(outdegree equations)}, \tag{5}$$

$$\sum_{ji \in A: j \neq i} x_{ji} = 1 \qquad \forall i \in V \qquad \text{(indegree equations)}, \tag{6}$$

$$\sum_{a \in C} x_a \leq |C| - 1 \qquad \forall C \in \mathcal{C} \qquad \text{(cycle inequalities)}, \tag{7}$$

$$0 \leq x_a \leq 1 \qquad \forall a \in A. \tag{8}$$

Denote by P_I the convex hull of the integral vectors in P.

(a) Show that that P_I is the *asymmetric traveling salesman polytope,* i.e., the convex hull of the Hamiltonian cycles in D.

(b) Derive the lifting coefficient for variable x_{13} in the inequality $x_{12} + x_{23} + x_{31} \leq 2$. I.e., find the largest coefficient α_{13} such that the inequality $x_{12} + x_{23} + x_{31} + \alpha_{13} x_{13} \leq 2$ is valid for P_I.

(c) Find all the sequential liftings of the 3-cycle inequality $x_{12} + x_{23} + x_{31} \leq 2$.

(d) Give a valid inequality for P_I that is a nonsequential lifting of $x_{12} + x_{23} + x_{31} \leq 2$. Hint: Recall the following inequalities, that are valid for P_I:

$$\sum_{ij \in A,\ i,j \in S} x_{ij} \leq |S| - 1 \qquad \forall \emptyset \subset S \subset V \qquad \text{(subtour elimination inequalities)}.$$

(e) Give an upper bound on the Chvátal rank of the nonsequential lifting obtained in (c).

4. **Nonlinear Optimization**

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function with Lipschitz continuous gradient, with Lipschitz constant L and a minimizer x^*. We know that for any $x, z \in \mathbb{R}^n$ that

$$f(z) \le f(x) + \nabla f(x)^T (z - x) + \frac{L}{2} \|z - x\|^2. \tag{9}$$

(a) Show (by minimizing both sides of (9) with respect to z) that for any $x \in R^n$ we have

$$f(x) - f(x^*) \ge \frac{1}{2L} \|\nabla f(x)\|^2.$$

(b) Prove the following *co-coercivity* property: For any $x, y \in \mathbb{R}^n$, we have

$$[\nabla f(x) - \nabla f(y)]^T (x - y) \ge \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2.$$

Hint: Apply part (a) to the following two functions:

$$h_x(z) := f(z) - \nabla f(x)^T z, \quad h_y(z) := f(z) - \nabla f(y)^T z.$$

(c) Suppose in addition to the properties of f given above, it is also *strongly* convex, with modulus of convexity $m > 0$. That is, for all $x, z \in \mathbb{R}^n$, we have

$$f(z) \ge f(x) + \nabla f(x)^T (z - x) + \frac{m}{2} \|z - x\|^2. \tag{10}$$

It is known that the function $q(x) := f(x) - \frac{m}{2}\|x\|^2$ is convex with $L - m$-Lipschitz continuous gradients. By applying the co-coercivity property of part (b) to this function q, show that the following property holds:

$$[\nabla f(x) - \nabla f(y)]^T (x - y) \ge \frac{mL}{m + L} \|x - y\|^2 + \frac{1}{m + L} \|\nabla f(x) - \nabla f(y)\|^2.$$