**Fall 2019 Qualifier Exam:**
**Decision Science and Operations Research**

**September 16, 2019**

## GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam (DSOR), your name, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered.
3. Return all answer books in the folder provided. Additional answer books are available if needed.
4. You are allowed *one* $8.5 \times 11$ sheet of paper with formulae.

## SPECIFIC INSTRUCTIONS:

You must answer 5 out of 6 questions.

## POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. **Poisson Processes**

   Ice cream production at Babcock Hall can be modeled as a Poisson process. Each day, the ice cream makers make three flavors of ice cream: vanilla, chocolate, and blue moon. They start all three recipes at the same time and make one batch of each kind per day.

   - The time to make a batch of vanilla or chocolate is distributed exponentially with a mean of 2.5 hours.
   - Blue moon is more difficult to make. The time to make a batch is distributed exponentially with a mean of 3 hours.

   (a) Find the probability that vanilla is the <u>first</u> batch to finish.

   (b) Find the probability that blue moon is the <u>last</u> batch to finish.

   (c) Lets say that there is an equipment failure and the ice cream makers can only make two flavors at a time on parallel. They decide to start with making vanilla and blue moon at the same time. When one finishes, they make chocolate on the equipment that is freed up. Find the probability that blue moon finishes <u>last</u>.

2. **Markov Processes**

A factory has two redundant pieces of equipment. When either one is working, the factory is operational. If at the beginning of a week both machines are working, then at the end of the week exactly one will be working with probability 0.3; neither will be working with probability 0.4. However, if only one machine is working at the beginning of a week, the probability that it will still be working at the end of the week is 0.4.

A repairman checks once a week to determine whether the factory is operational. If not, it takes him a week to fix both broken machines.

(a) What fraction of all weeks is spent in repair (in the long run)?

(b) How many repairs do we expect will be performed on average during a period of $n$ weeks (in the long run)?

(c) Why might your answers to parts (a) and (b) above not apply if the machine starts in perfect working order, and we are interested in its performance during the first $n$ weeks (rather than in the long run)? What methods could be used to characterize the short run behavior of the machine?

(d) Would this system still be Markovian if the transition probabilities were uncertain (e.g., described by probability distributions) rather than known constants? Why or why not? Explain your reasoning.

3. **Simulation**

(a) Consider an unbounded continuous random variable $X$ having cumulative distribution function (cdf) $F$. Suppose you wish to generate observations of a random variable $Y$, which has the distribution of $X$ but is *truncated* to the interval $[a, b]$. Specifically, $Y$ has the cdf given by $F_Y(y) = 0$ for $y < a$,

$$F_Y(y) = \frac{F(y) - F(a)}{F(b) - F(a)}, \quad a \leq y \leq b,$$

and $F_Y(y) = 1$ for $y > b$. Assume that you have an algorithm that can efficiently generate independent and identically distributed random outcomes having the distribution of $X$.

i. Derive an acceptance/rejection type algorithm for generating a random variate having distribution of $Y$ using the method for generating observations of $X$, and demonstrate that the method is correct. (I.e., state the algorithm, and show that the output has the correct distribution.)

ii. What is the expected number iterations the acceptance/rejection method requires?

iii. Consider the following routine:
Step 1. Generate $U$ uniformly in $[0, 1]$ and let $V = F(a) + U[F(b) - F(a)]$.
Step 2. Let $Y = F^{-1}(V)$.
Show that the output of this routine has the distribution of $Y$.

iv. Discuss the potential trade-off in computation effort between the algorithm in part (i) and that in part (iii).

(b) Consider a nonterminating simulation of a service queueing system, producing a sequence of observed waiting times $X_1, X_2, X_3, \ldots$. You are interested in estimating the value of $\mu = E[X_i]$, for the system once it is in steady state (at which point the value $E[X_i]$ is independent of $i$). You have estimated that the system appears to be roughly in steady state after $L$ observations.

i. Describe how you would estimate and construct a confidence interval for $\mu$ using the replication/deletion approach. (Your description should include mathematical descriptions as needed to be clear.)

ii. Describe how you would estimate and construct a confidence interval for $\mu$ using the batch means approach. (Your description should include mathematical descriptions as needed to be clear.)

iii. Assuming you simulate roughly the same number of observations of $X_i$ in the two approaches, describe one potential advantage of the replication/deletion approach over the batch means approach, and vice versa one potential advantage of the batch means approach over the replication/deletion approach.

4. **Optimization Modeling**

You have been tasked with determining how to best pickup crates of books from five different library locations and return them all to the central library. In this model, the central library is located at the origin of an $xy$-grid, and the $(x, y)$ coordinates of the other 5 locations are given in the table below, along with how many crates of books are at each location.

| **Location** | **# Crates** | **$x$-coordinate** | **$y$-coordinate** |
|---|---|---|---|
| A | 9 | -2 | 1 |
| B | 6 | 2 | 2 |
| C | 4 | 3 | 0 |
| D | 3 | 2 | -2 |
| E | 5 | -1 | -1 |

For simplicity, we will measure the distances between cities using the "Manhattan"-metric (or the $\ell_1$ distance) where the bookmobile can only go north-south or east-west on the grid. For example, the distance between locations $A$ and $B$ is

$$d_{AB} = |-2-2| + |1-2| = 5.$$

The bookmobile starts the day at the central library and can carry at most 12 crates of books. Further, if the bookmobile goes to a location, it *must* take all of the books at that location. Thus, if the bookmobile goes to location $A$, it either has to return to the central library or go to location $D$, all other locations have too many books.

(a) Create an integer programming instance that will minimize the total distance the bookmobile has to travel to return all of the crates of books to the library. (Note that just solving the problem by inspection, or giving a model that relies significantly on inspection, will not get very much credit. We are looking for a model that would work even if there are many more locations.)

(b) Modify your model from item (a) to maximize the number of crates of books returned to the library if the bookmobile can only make two trips.

(c) Take the dual of the linear programming relaxation of the model that you built in item (a).

(d) Now suppose you are given a set of $N$ locations with coordinates $(x_i, y_i)$ and a number of crates $w_i$ $\forall i = 1, 2 \ldots, N$. The capacity of the bookmobile is $Q$. Describe how to modify your instance from item (a) for this general case. What is the size of your instance? If $N > 100$, how might you go about solving an instance of the model?

5. **Linear Programming**

You may use the following version of Farkas' Lemma in this problem:

**Theorem 1** (Farkas' Lemma)**.** *For a linear inequality system* $\Sigma\colon \{Ax \le b\}$*, either* $Ax \le b$ *is feasible, or there exists* $y \in \mathbb{R}^m$ *such that* $y \ge 0$*,* $y'A = 0$ *and* $y'b = -1$*, but not both.*

Let $\Sigma : \{Ax \le b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, be an infeasible system of linear inequalities. An infeasible subsystem $\Sigma'$ of $\Sigma$ is an *Irreducible Infeasible Subsystem (IIS)* if every proper subsystem of $\Sigma'$ is feasible. By answering the question in this problem you will prove the following result, that relates the IISs of $\Sigma$ to the extreme points of a given alternative polyhedron. Recall that the support of a vector $x$, denoted by $\mathrm{supp}(x)$, is the set of indices of the nonzero components of $x$.

**Theorem 2.** *The IISs of* $\Sigma$ *are in one-to-one correspondence with the extreme points of the polyhedron*

$$P = \{y \in \mathbb{R}^m : y \ge 0, y'A = 0, y'b = -1\}.$$

*Precisely, the support of any extreme point of* $P$ *indexes the inequalities of an IIS.*

(a) Consider a polyhedron $P \subseteq \mathbb{R}^n$. Give the definitions of: $(i)$ extreme point; $(ii)$ vertex; and $(iii)$ basic feasible solution.

(b) The three definitions above are equivalent. State a necessary and sufficient condition for a polyhedron to contain at least a vertex.

(c) Let $\Sigma_1 : \{A_1x \le b_1\}$ be an IIS of $\Sigma$, and suppose wlog that $A_1$ and $b_1$ consist of the first $m_1$ rows of $A$ and $b$, respectively. Let $P_1 = \{w \in \mathbb{R}^{m_1} : w \ge 0, w'A_1 = 0, w'b_1 = -1\}$. Show that there exists a vertex $y$ of $P$ such that $\mathrm{supp}(y) = \{1, \dots, m_1\}$ by following the steps below:

   i. Prove that $P_1$ is nonempty, i.e. it contains a point $\bar{w}$.
   ii. Prove that all vectors in $P_1$ are strictly positive, i.e. $w > 0\ \forall\, w \in P_1$.
   iii. Prove that $P_1$ contains a vertex, and that $\bar{w}$ is a vertex of $P_1$. In fact, $P_1 = \{\bar{w}\}$.
   iv. Construct a vertex $y$ of $P$ such that $\mathrm{supp}(y) = \{1, \dots, m_1\}$.

(d) Let $y$ be a vertex of $P$ and assume wlog that $\mathrm{supp}(y) = \{1, \dots, m_1\}$. Show that $\Sigma_1 : \{A_1x \le b_1\}$ is an IIS of $\Sigma$, where $A_1$ and $b_1$ consist of the first $m_1$ rows of $A$ and $b$, respectively. *Hint: First, prove that* $\Sigma_1$ *is an infeasible subsystem of* $\Sigma$*. Then to prove that* $\Sigma_1$ *is* irreducible *proceed by contradiction: assuming it is reducible* $(i)$ *show that there exists* $u \in P$ *with supp*$(u) \subset s(y)$*;* $(ii)$ *prove that* $y - u$ *is a feasible direction at* $y$*, and use it to show that* $y$ *is not an extreme point of* $P$*.*

6. **Integer Optimization**

For every $j \in \{0, \ldots, n\}$, let $S_j$ be the set of vectors $x \in \mathbb{R}^n$ such that $j$ components of $x$ are $1/2$ and the remaining $n - j$ components are equal to 0 or 1. For example, the set $S_1^2$ is defined by

$$S_1^2 := \{(1/2, 0), (1/2, 1), (0, 1/2), (1, 1/2)\}.$$

(a) (1 point) Show that, for every $j \in \{0, \ldots, n-1\}$, any vector $v \in S_{j+1}$ is the convex combination of two vectors in $S_j$.

(b) Given $j \in \{0, \ldots, n-1\}$, consider any $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$ such that $\pi x < \pi_0$ for every $x \in S_j$ (note the strict inequality). Show that $\pi x \leq \pi_0 - 1$ for every $x \in S_{j+1}$. To prove this result, let $v \in S_{j+1}$ and consider the following two cases:

(b1) (1 point) First prove $\pi v \leq \pi_0 - 1$ when $\pi v \in \mathbb{Z}$. [Hint: Use (a).]

(b2) (3 points) Then prove $\pi v \leq \pi_0 - 1$ when $\pi v \notin \mathbb{Z}$. [Hint: Without loss of generality, assume that $v_1 = 1/2$ and $\pi_1 \neq 0$. Then write $v$ as the convex combination of two vectors $v^1, v^2 \in S_j$. Show that, for $i = 1, 2$, each of the line segments $[v, v^i]$ contains a point $\tilde{v}^i$ such that $\pi \tilde{v}_i \in \mathbb{Z}$. Deduce that $\pi v \leq \pi_0 - 1$.]

(c) For any $n \geq 1$, let

$$P := \left\{ x \in \mathbb{R}^n : 0 \leq x \leq 1, \quad \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq 1/2 \quad \forall J \subseteq \{1, 2, \ldots, n\} \right\}.$$

(c1) (1 point) Show that $P \cap \mathbb{Z}^n = \emptyset$.

(c2) (1 point) Show that $S_1 \subset P$.

(c3) (3 points) Use what you have proven so far to show that the Chvátal rank of $P$ is at least $n$. *Hint: Use the observation that every Chvátal cut for a polyhedron $Q$ can be written in the form $\pi x \leq \pi_0 - 1$, where $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$ and $\pi x < \pi_0$ for every $x \in Q$.*