Fall 2015 Qualifier Exam:
Decision Science and Operations Research

September 21, 2015

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam (DSOR), your name, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered.
3. Return all answer books in the folder provided. Additional answer books are available if needed.
4. You are allowed *one* 8.5×11 sheet of paper with formulae.

SPECIFIC INSTRUCTIONS:

You must answer 5 out of 6 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Given rational numbers $c_j, j = 1, \ldots, n$, $b_i, i = 1 \ldots, m$, and $a_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$, the primal form of a linear program is

$$\max \quad \sum_{j=1}^{n} c_j x_j \tag{PLP}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1, \ldots, m$$

$$x_j \geq 0 \quad \forall j = 1, \ldots, n.$$

   (a) Write the dual linear program of (PLP).

   (b) State the weak duality theorem.

   (c) Prove the weak duality theorem you stated in part (b).

   (d) State the strong duality theorem.

   (e) State the complementary slackness theorem.

   (f) Use the strong duality theorem to prove the complementary slackness theorem.

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2. Lazy Lane is a straight highway that extends for 10km. There are n people who live on Lazy Lane at locations $p_j, j = 1, \ldots, n$, where p_j is the distance in km from the beginning of Lazy Lane to the j^{th} person's house. There is one giant alarm clock on Lazy Lane that wakes up all of its residents at exactly the same time, at which time they immediately start walking to the bus stop to go to work. Person j walks to the bus stop at rate r_j and needs to arrive to the bus stop by time $t_j, i = 1, \ldots, n$. The alarm clock goes off exactly once per day. The Lazy Lane Bus Company (LLBC) is placing a single bus stop on Lazy Lane.

   (a) LLBC would like to know where to place the bus stop so as to allow the residents of Lazy Lane to sleep as late as possible. Formulate this problem as a linear program.

   (b) Now suppose that LLBC is going to place m bus stops on Lazy Lane. Person j will get to work on time if (s)he gets to *any* of the bus stops by time t_j. LoserLinderoth, Lord of Lazy Lane, loves Linear Programming. He claims that you can also model this decision problem as a linear program? Is LoserLinderoth correct? If so, model the problem as a linear program. If not, explain why.

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3. Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, we denote by $P_I := \text{conv.hull}(P \cap \mathbb{Z}^n)$ the integer hull of P, and by P' the Chvátal closure of P. Formally,

$$P' := \{x \in \mathbb{R}^n : ax \leq \lfloor \beta \rfloor, \text{ for every } ax \leq \beta \text{ valid for } P \text{ with } a \in \mathbb{Z}^n\}.$$

Moreover for every $i \in \mathbb{N}$ we denote by $P^{(i)}$ the i-th Chvátal closure of P, i.e. $P^{(0)} := P$ and $P^{(i)} := (P^{(i-1)})'$ for $i \geq 1$.

Prove or disprove:

(a) For every two polyhedra $P, Q \subseteq \mathbb{R}^n$ with $P \subseteq Q$, $P_I \subseteq Q_I$.

(b) For every two polyhedra $P, Q \subseteq \mathbb{R}^n$ with $P \subseteq Q$, $P' \subseteq Q'$.

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, we denote with $\mathcal{R}(A, b)$ the family of the polyhedral relaxations of P that consist of the intersection of the half-spaces corresponding to a family of linearly independent inequalities $\bar{A}x \leq \bar{b}$ of the system $Ax \leq b$. Prove or disprove:

(c) $P = \cap_{R \in \mathcal{R}(A,b)} R$;

(d) $P' = \cap_{R \in \mathcal{R}(A,b)} R'$;

(e) $P_I = \cap_{R \in \mathcal{R}(A,b)} R_I$.

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4. Due to start-up problems, a pump in a nuclear-power plant has a 0.5 chance of failing during its first hour of operation. If the pump survives the first hour of operation, then the probability of surviving each succeeding hour of operation is 0.9.

The maintenance crew checks the condition of the pump at the end of each hour. If the pump is not working at the beginning of any given hour, there is a 0.7 chance that the crew will find the fault and have the machine ready to turn on again at the end of the hour, and a 0.3 chance that the pump will still not be working at the end of the hour.

(a) Formulate as a discrete-time Markov process, and draw the transition diagram. (Hint: Your process should have three states.)

(b) Is this process periodic? Explain why or why not.

(c) Determine the limiting-state probabilities of this Markov process. According to those results, what fraction of all hours is spent in maintenance (in the long run)?

(d) For a randomly selected hour, what is the joint probability that the pump will be not working at the beginning of that hour **and** transition to a working state by the end of the hour? What is the overall probability that the Markov process will transition from one state to another in a randomly selected hour?

(e) What is the distribution for the number of hours that the pump spends in maintenance before returning to a working state, for any given visit to the maintenance state?

(f) Now, assume that the probability that the maintenance crew will find the fault in the nth hour of a failure is not 0.7, but $0.7 + 0.3(1 - e^{-(n-1)})$, so that a repair becomes more and more likely the longer the pump has not been working. Can the resulting problem still be represented as a three-state Markov process? If not, please describe at least one way in which the problem could be modeled as a stochastic process (rather than just simulating it).

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5. A local government facility maintains its power systems. Failures to the main system occur every F days, where F is exponentially distributed with an average of $1/\lambda = 100$ days. When such a failure occurs, the facility uses its backup generator to maintain power and immediately order a repair. The backup generator lasts for B days, where B is exponentially distributed with mean $1/\gamma = 2$, and the repair occurs after R days, where R is exponentially distributed with mean $1/\mu = 1.5$ days. A repair costs \$1500. If the repair does not occur before the backup generator fails, then the facility incurs a total failure cost of $\$14,000$

   (a) Find the probability that the repair occurs before the backup generator fails.

   (b) Find the *expected* cost, including possible repair and total failure costs.

   (c) Suppose an expedited repair can be ordered at a cost of C, where the time until the expedited repair E is exponential with mean $1/\beta = 0.5$ days. Find the most you would be willing to pay for the expedited repair.

   (d) Suppose that expedited repairs are no longer available, so the facility purchases a second backup generator that is used if the first backup generator fails. Both generators have independent and identically distributed failure times. The repair order is placed when a failure to the main system occurs (same as before), but now two backup generators must fail before a total failure occurs. Redo (b) in this scenario.

   (e) Refer to (d). Find the probability that the expedited repair arrives while the second backup generator is still working.

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6. A large parts supplier to the automobile industry wants to determine the proper number of dedicated retriever cranes and total cost for a planned Automated Storage and Retrieval System (AS/RS). [Each retriever crane only services one aisle of the AS/RS.] Some design parameters and assumptions are as follows:

   - The company needs 6,000 storage openings (52"w x 36"h x 48"d) in the AS/RS, and each opening costs $250.
   - Retriever cranes can travel at 300 ft/min (horizontal) and 150 ft/min (vertical).
   - The time to position and either place a pallet load into or take a load out of a storage opening is 4 seconds. The same positioning and movement times apply to load transfers at the input/output station of the AS/RS.
   - The maximum height of a storage rack is 65 feet.
   - The maximum length of any aisle is 500 feet.
   - Each retriever crane costs $225,000 with an annual maintenance cost of $30,000.
   - The company operates two 8-hours shifts per day.
   - The cost of a load waiting for service is estimated to be $35.00/hour.

   You have been asked by a company to develop a simulation model for this AS/RS before they purchase the AS/RS. Before they issue a purchase order to you for the cost of developing the model, they want you to address the following questions:

   (a) How would you model the arrival process for demands for cranes services (to either store and/or retrieve a load)? Address how the data would be collected and modeled.

   (b) How would you model the service process distribution given the physical size the planned AS/RS?

   (c) How would you use the simulation model to determine how many retriever cranes the company should plan for given that they wish to minimize the cost of the system?

   (d) What considerations and assumptions would you need to include?