Fall 2016 Qualifier Exam:
OPTIMIZATION

September 19, 2016

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer all 4 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Suppose that

$$g(x) := \min_y c^T y \text{ subject to } Ay = b + Dx,\ y \geq 0$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, c is a given p vector, A a given $m \times p$ matrix, b is a given m vector and D a given $m \times n$ matrix. Assume that

$$0 = \min_y c^T y \text{ subject to } Ay = 0,\ y \geq 0$$

and

$$\{z : z = Ay,\ y \geq 0\} = \mathbf{R}^m.$$

   (a) Give an example of a matrix A and a vector c that satisfy the assumptions.

   (b) For a given x, write down the dual of the problem defining $g(x)$.

   (c) Under the assumptions, show that $g(x)$ is finite for all $x \in \mathbf{R}^n$.

   (d) Under the assumptions, show that g is convex on $\mathbf{R}^n$.

   (e) What is another property that g has?

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2. Let $D = (V, A)$ be a directed graph, and let $w : A \to \mathbb{R}$ be a weight vector. The weight of a subset B of A is defined as $w(B) := \sum_{a \in B} w_a$. Consider the problem of finding a maximum-weight subset $B \subseteq A$ such that no node of V is at the same time the head of an arc in B and the tail of another arc in B.

   (a) Formulate this problem as a $0, 1$ linear program.

   (b) Is the polyhedron defined by the natural linear programming relaxation of your $0, 1$ linear set integral? Provide a justification of your answer either way.

   (c) If the answer to part (b) is "no", give an additional class of inequalities, which is not implied by the inequalities of your natural linear programming relaxation, but which is valid for all its $0, 1$ solutions. (Hint: Think about certain directed cycles in D.)

3. The city of Sol, wanting to stay true to its name, is assessing the feasibility of using solar panels to provide power for its inhabitants during the warm summer months, when air conditioning costs are the highest. A multi-year survey of the inhabitants' energy habits as well as the hourly solar availability revealed the following aggregate data representing a typical summer day:

Hour of day:	1	2	3	$\cdots$	24
Hourly demand (MWh):	d_1	d_2	d_3	$\cdots$	d_{24}
Solar intensity:	s_1	s_2	s_3	$\cdots$	s_{24}

Unfortunately, demand doesn't always align itself with the solar cycle. For example, there is less sun in the evening, when the demand for energy is highest. The strategy is as follows:

- Purchase some number (N_p) of solar panels. Each solar panel costs C_p dollars and provides a maximum amount of MWh equal to the solar intensity. For example, 150 solar panels can provide up to $150s_1$ MWh during the first hour. Note that this is the *maximum* yield of the panels. We can always have the panels produce less if need be.
- Purchase some number (N_b) of batteries. Each battery costs C_b dollars and can be used to store up to 1 MWh of power, which may be used at a later time. Each battery has an hourly efficiency of 98%, so each hour the batteries lose 2% of their total stored energy. It is required that the batteries end the day with the same level of charge that they had when the day started.

Now, the problems:

(a) Write an optimization model (define and explain the variables, constraints, and objective) that would solve the problem of figuring out how many solar panels N_p and batteries N_b should be purchased so that total cost is minimized and the city will generate enough power to meet the hourly demands with a 5% buffer. In other words, we would like to be able to provide up to $1.05d_t$ of power at time t. What sort of optimization problem is this?

(b) Sol followed your recommendation and purchased N_b batteries and N_p solar panels. To reduce wear and tear on the batteries, Sol would like to operate their system in a way that minimizes the number of times the batteries are charged or discharged. This time, we will run our system with no buffer (we must exactly meet the hourly energy demands). How can this be accomplished? What sort of optimization problem is this?

4. Consider solving the problem $\min_{x\in\mathbb{R}^n} f(x)$, where f is Lipschitz continuously differentiable, with $\|\nabla f(x) - \nabla f(y)\|_2 \le L\|x-y\|_2$ for all x, y. As a consequence of Taylor's theorem, we have for all $x, d \in \mathbb{R}^n$ and all $\alpha > 0$ that

$$f(x+\alpha d) \le f(x) + \alpha \nabla f(x)^T d + \frac{1}{2} L\alpha^2 \|d\|^2. \quad \textbf{(Qbound)}$$

Suppose we use a line-search method to minimize f, with steps of the form $x_{k+1} = x_k + \alpha_k d_k$.

(a) Consider the Gauss-Southwell choice for search direction $d_k = -[\nabla f(x_k)]_{i_k} e_{i_k}$, where e_{i_k} is the unit vector $(0,\dots,0,1,0,\dots,0)^T$ with the 1 in position i_k, where

$$i_k := \arg \max_{i=1,2,\dots,n} |[\nabla f(x_k)]_i|.$$

Find positive values of $\bar{\epsilon}$, γ_1, and γ_2 such that this d_k satisfies conditions

$$-d_k^T \nabla f(x_k) \ge \bar{\epsilon}\|\nabla f(x_k)\|_2 \|d_k\|_2, \quad \gamma_1 \|\nabla f(x_k)\|_2 \le \|d_k\|_2 \le \gamma_2 \|\nabla f(x_k)\|_2.$$

(b) Suppose the search direction d_k is a descent direction, satisfying the conditions

$$-d_k^T \nabla f(x_k) \ge \bar{\epsilon}\|\nabla f(x_k)\|_2 \|d_k\|_2, \quad \|d_k\|_2 \ge \gamma_1 \|\nabla f(x_k)\|_2.$$

for positive $\bar{\epsilon}$ and γ_1. Suppose that we use a backtracking procedure to select α_k, where we try in turn $\alpha_k = \bar{\alpha}, \bar{\alpha}/2, \bar{\alpha}/4, \dots,$ for some $\bar{\alpha} > 0$, stopping when the following sufficient decrease condition is satisfied:

$$f(x_k + \alpha_k d_k) \le f(x_k) + c_1 \alpha_k d_k^T \nabla f(x_k), \quad \textbf{(SuffDecr)}$$

for some constant $c_1 \in (0,1)$. Find a value $\Delta_1 > 0$ such that when no backtracking is required (that is, the value $\alpha_k = \bar{\alpha}$ satisfies condition **(SuffDecr)**), we have

$$f(x_{k+1}) \le f(x_k) - \Delta_1 \|\nabla f(x_k)\|_2^2,$$

(c) Consider the same setup as in (b), but suppose now that bactracking is required, so that $\alpha_k < \bar{\alpha}$. That is, condition **(SuffDecr)** is satisfied by α_k but is violated when α_k is replaced by $2\alpha_k$, which was the previous value tried in the backtracking process. Find a positive lower bound on α_k.

(d) Considering again the same setup as in (b) and (c), and using the lower bound derived in (c), find a value $\Delta_2 > 0$ such that when backtracking is required, we have

$$f(x_{k+1}) \le f(x_k) - \Delta_2 \|\nabla f(x_k)\|_2^2,$$

Thus, find a value $\Delta > 0$ such that regardless of whether backtracking is needed or not, we have

$$f(x_{k+1}) \le f(x_k) - \Delta \|\nabla f(x_k)\|_2^2.$$