Fall 2012 Qualifier Exam:  
Decision Science and Operations Research  

September 24, 2012  

GENERAL INSTRUCTIONS:  

1. Answer each question in a separate book.  

2. Indicate on the cover of each book the area of the exam (DSOR), your name, and the question answered in that book. On one of your books list the numbers of all the questions answered.  

3. Return all answer books in the folder provided. Additional answer books are available if needed.  

SPECIFIC INSTRUCTIONS:  

You must answer 5 out of 6 questions.  

POLICY ON MISPRINTS AND AMBIGUITIES:  

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.
1. Consider an $M/M/1$ queuing system with an arrival rate of $\lambda$ and a service rate of $\mu$, where the customers are impatient. A customer who finds the server busy will wait for a time that is exponentially distributed with mean $1/\beta$, and leave without service if the service has not begun by that time.

(a) Construct a Markov chain and determine the transition-rate matrix for this queuing system.

(b) Determine the average queue length and mean waiting time of an arbitrary customer in this queuing system.

(c) How would you analyze this problem and determine answers to a) and b) if each customer was prepared to spend at most a fixed deterministic time of $1/\beta$ in the queue (instead of an exponential amount of time)?

2. Suppose that random variables $Y_1, Y_2, Y_3, \ldots$ are the observations of a simulation run.

(a) If $E_s$ is an event that occurs with probability $1 - \alpha_s$ at the $s$-th observation, for $s = 1, 2, 3, \ldots$ prove that

$$P(\cap_{s=1}^{k} E_s) \geq 1 - \sum_{s=1}^{k} \alpha_s.$$  

You should not assume that the $E_s$ are independent.

(b) If $Y_1, Y_2, Y_3, \ldots$ is a covariance-stationary process, then we know that $\text{Var}(Y_i) = \sigma^2$ and $\text{Cov}(Y_i, Y_{i+j}) = c_j$. Let $\bar{Y}_m = \frac{1}{m} \sum_{i=1}^{m} Y_i$. Show that if $\sum_{i=1}^{\infty} c_j < \infty$, then as $m \to \infty$, $\text{Var}(\bar{Y}_m) \to 0$.

3. You are developing a discrete-event simulation model of a production system consisting of a tandem production line with exponential service times. External arrivals to this production line constitute a Poisson process with parameter $\lambda$

(a) Describe an algorithm that would effectively simulate Poisson arrivals to this production system.

(b) How would you adapt your algorithm to generate an arrival process with inter-arrival times that are independent and identically distributed in general (i.e., not necessarily Poisson)? Comment on the challenges, and the assumptions you would need to make.

(c) How would you adapt your algorithm to generate arrival processes that are non-stationary? Comment on the challenges, and the assumptions you would need to make.
4. In this problem, we will consider a production/distribution problem with a set $J$ of customers whose (random) demand $d_j(\xi)$ must be met from a set of facilities $I$. Suppose that the random demand for customer $j$ comes from a discrete distribution; namely, that the demand of customer $j$ in scenario $s \in S$ is $d_{js}$ with probability $p_s$, for a finite set of scenarios $S$. Opening a facility $i \in I$ costs a fixed amount $f_i$, and shipping one unit from $i \in I$ to $j \in J$ costs an amount $c_{ij}$. We must decide which facilities to open before we know the demand from each customer, but the distribution decisions are made after knowing the customer demand. You may assume that each facility has a maximum capacity $u_i$.

(a) If we are not able to meet customer demand from open facilities in a scenario, then we must compensate our customers an amount $\lambda$ times the shortage. Write an optimization model whose solution will minimize the total expected cost. Be sure to clearly define your decision variables.

(b) We would like to impose the constraint that the probability that all customers get their demand met to be at least $1 - \epsilon$. Adapt your model from part (a) to model this condition.

5. Consider the following convex quadratic program with a single equality constraint, non-negativity constraints, and a diagonal Hessian:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x \quad \text{subject to} \quad a^T x = 1, \quad x \geq 0, \quad (1)$$

where $a \in \mathbb{R}^n$ is a vector with all positive entries, and $Q$ is a diagonal matrix with all positive diagonal entries.

(a) Suppose we drop the bounds $x \geq 0$ from the formulation (1). Write down the KKT conditions for the resulting simplified problem, and use them to deduce the solution $x$ in closed form.

(b) Returning to the full problem (1), write down the KKT conditions, denoting the Lagrange multiplier for the constraint $a^T x = 1$ by $\lambda$.

(c) Fixing the value of $\lambda$ in these KKT conditions, find the value of $x_i$, $i = 1, 2, \ldots, n$ that satisfies these conditions as an explicit function of $\lambda$. (Use the notation $x_i(\lambda)$, $i = 1, 2, \ldots, n$ to denote these values.)

(d) Show that the function $t : \mathbb{R} \to \mathbb{R}$ defined by

$$t(\lambda) = a^T x(\lambda) - 1 = \sum_{i=1}^{n} a_i x_i(\lambda) - 1$$

is a monotonic piecewise linear function of $\lambda$, and identify the breakpoints of this function (the points where the slope changes discontinuously).
6. The Christmas board game “22” involves a board with 13 holes and 13 pegs which fit in the holes. The pegs are numbered from 1 to 13. Holes are situated at the 12 intersection points on a six-pointed star and in the center of the star. To play the game, a peg is inserted in each hole. A winning configuration is one in which the sum of values for each of the six outer triangles sums to 22. Here, for example, is a winning assignment:

In your solution, use the following indexing scheme to reference the game board holes:

(a) Determine which variables are needed to provide a solution to the game?
(b) Define a mapping $H(t)$ that provides the subset of “holes” used in triangle $t$ and use this to write down the “22” constraint. Note that $t$ will range from 1 to 6, indicating each of the “outer” triangles.
(c) Write the full mathematical (or GAMS) model which finds a solution to the game.
(d) Suppose this model is solved for one solution. Determine an additional constraint that would eliminate just this solution, and enable the model to be rerun to find another solution.
(e) What techniques could you use to remove “equivalent solutions” from within your model search? Provide constraints that remove such “symmetries” from your search.
(f) Write pseudo-code (GAMS or similar for example) that shows the sequence of model solves that will find all solutions to the game.